Onset of dynamo action in an axisymmetric flow

A. Tilgner
Institute of Geophysics, University of Göttingen, Herzberger Landstraße 180, 37075 Göttingen, Germany
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Peffley, Cawthorne, and Lathrop [Phys. Rev. E **61**, 5287 (2000)] have reported on an experiment using liquid sodium, which studies the approach toward a self-generating dynamo. Their results challenge the traditional views of kinematic dynamo theory because (i) the modes of the magnetic field with the smallest decay rates appear to be nearly axisymmetric and (ii) the observed decay rates vary spatially. This report shows how these observations can be reconciled with kinematic dynamo theory.

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As part of an international effort to reproduce the dynamo effect on a laboratory scale (see Ref. [1] for a review) Peffley *et al.* [2] have recently started an experimental investigation of a class of flows originally studied numerically by Dudley and James [3]. Similar flows are currently under study by other groups as well [4,5]. These flows exist in a sphere and are axisymmetric with a velocity field $\boldsymbol{v}$ represented in spherical polar coordinates (r,ϑ,φ) by

$$\boldsymbol{v}=\epsilon\nabla\times\nabla\times f_p(r)P_2(\cos\vartheta)\hat{\boldsymbol{r}}+\nabla\times f_t(r)P_2(\cos\vartheta)\hat{\boldsymbol{r}}, \tag{1}$$

where $P_2(\cos\vartheta)$ is the second Legendre polynomial and $\hat{\boldsymbol{r}}$ the unit position vector. For suitable choices of the coefficient ϵ and the radial profiles $f_p(r)$ and $f_t(r)$, there exist solutions growing in time of the induction equation for the magnetic field $\boldsymbol{B}(\boldsymbol{r},t)$:

$$\frac{\partial}{\partial t}\boldsymbol{B}+\mathrm{Rm}\nabla\times(\boldsymbol{B}\times\boldsymbol{v})=\nabla^2\boldsymbol{B}. \tag{2}$$

All variables in Eqs. (1) and (2) are nondimensional. Rm is the magnetic Reynolds number given by $\mathrm{Rm}=VL/\eta$, where V and L are the velocity and length scales of the flow, and η is the magnetic diffusivity of the conducting fluid.

If $\boldsymbol{v}$ is independent of time, solutions of Eq. (2) can be expressed as a superposition of eigenmodes that have the form $\boldsymbol{B}=\boldsymbol{B}_0(\boldsymbol{r})e^{\lambda t}$ with the growth rate λ. According to Cowling's theorem there is no axisymmetric $\boldsymbol{B}_0(\boldsymbol{r})$ for which $\mathrm{Re}\{\lambda\}>0$.

An awkward linguistic detail bedevils every presentation on kinematic dynamo theory. The word "decay rate" designates $-\lambda$. Suppose two different modes have growth rates λ_1 and λ_2. Mode 1 is then said to have the larger decay rate if $\mathrm{Re}\{\lambda_1\}<\mathrm{Re}\{\lambda_2\}$, i.e., if mode 1 has the smaller growth rate.

The experiment in Ref. [2] approximates a flow as described by Eq. (1) by driving liquid sodium in a sphere with two counter-rotating propellers. The axisymmetry of the theoretical model is broken in the experiment by turbulent fluctuations. Even the time averaged flow is not axisymmetric because of baffles running from pole to equator in each hemisphere, which are intended to promote the poloidal component of the velocity field. There is no quantitative information available about the velocity field but the small size of the baffles (5% of the sphere diameter) suggests that the nonaxisymmetric components should be small compared to the axisymmetric component of the time averaged velocity field.

The Rm reached in the experiment is insufficient to obtain self-excitation of the magnetic field. The approach to the critical Rm is diagnosed by applying an external magnetic field either parallel or perpendicular to the propeller axes and by observing the decay of $\boldsymbol{B}$ after the external field has been switched off. The radial component of the magnetic field is measured by an array of Hall probes placed on a ring coaxial with the axis of symmetry of the flow. The decay rates of the nonaxisymmetric fields excited by an external field perpendicular to the axis of symmetry increase with increasing Rm (Fig. 6 of Ref. [2]). On the other hand, a field parallel to the shafts excites predominantly axisymmetric modes. The decay rate of these modes decreases with increasing Rm up to the highest Rm reached in the experiment. Extrapolation of the experimental curve to higher Rm predicts that the growth rate will become positive at sufficiently high Rm (around 160). Peffley *et al.* concluded that nonaxisymmetric components in their flow must be essential to the dynamo mechanism because an axisymmetric velocity field generates a dynamo field without any axisymmetric component according to kinematic dynamo theory.

The second experimental finding of relevance here is the apparent spatial variability of the decay rates. Surprisingly, different decay rates of the magnetic field are measured at different locations. It is not surprising that different decay rates are measured from pulse to pulse if the external magnetic pulse is repeatedly applied to a turbulent flow. However, if the decaying field following a single pulse corresponds to the eigenmode of Eq. (2) with the smallest decay rate, one expects to find one single decay rate at all positions after an initial transient. The value of that decay rate depends on the actual shape of the (time dependent) velocity field during the decay.

Sweet *et al.* [6] have attempted to interpret these results in terms of blowout bifurcations. But the computations in Ref. [6] are only for magnetic Reynolds numbers close to critical, whereas the experiment has been far below critical (see below). In addition, the blowout bifurcation scenario does not make any predictions on spatial correlations.

It will be shown in this paper that the experimental data are nonetheless compatible with the assumption that global

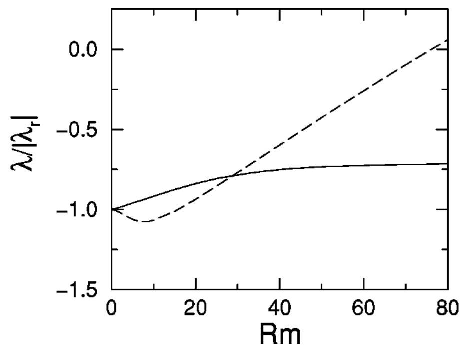

FIG. 1. The growth rate λ, normalized with the free decay rate λ_r for $m=0$ (continuous line) and $m=1$ (dashed line). The velocity field is given by Eq. (1) with $\epsilon=0.25$ and the radial functions (3).

magnetic eigenmodes of an axisymmetric velocity field are observed. A model based on kinematic dynamo theory reproduces the experimental findings if one assumes that a superposition of eigenmodes is excited by the combined effect of the external field and turbulent fluctuations.

In order to make the connection with kinematic dynamos, calculations similar to those of Dudley and James [3] have been repeated for magnetic fields with azimuthal wave numbers m equal to 0, 1, 2, and 3. Components with different m decouple in an axisymmetric flow and can therefore be investigated separately. The same numerical method as in Ref. [7] has been used to integrate the induction equation (2) in a spherical shell with inner and outer radii r_i and r_o such that $r_i=1/9$ and $r_o=r_i+1$. The inner core and the outer region are assumed to be vacuum. The presence of the small inner core (which occupies 0.1% of the volume enclosed by the outer sphere) facilitates the numerical computation and has no noteworthy influence on the results. The qualitative results of interest here do not depend on the particular choice of the radial profiles in Eq. (1). Figures 1 and 2 have been computed for

$$f_t(r)=\rho(1-\rho)\sin(\pi\rho), \quad f_p(r)=\rho f_t(r), \quad \rho=(r-r_i). \tag{3}$$

Figures 1 and 2 are for $\epsilon=0.25$ and 0.1, respectively. The case $\epsilon=0.1$ is not a dynamo up to Rm=300. The growth rates in these examples are all purely real. In the following, it is understood that the eigenvalues λ are real numbers.

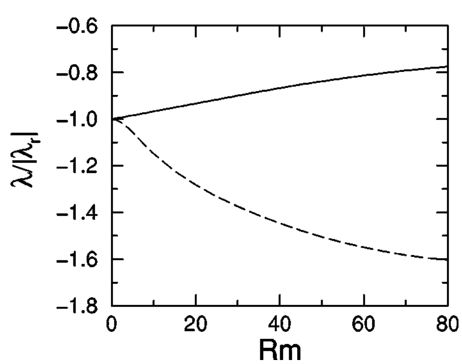

FIG. 2. Same as Fig. 1 but for $\epsilon=0.1$.

While Cowling's theorem excludes axisymmetric dynamo fields, it makes no statement on the behavior of axisymmetric fields below onset. In Figs. 1 and 2 and all other investigated cases, the axisymmetric mode has a smaller decay rate than the $m=1$ mode for small Rm. The growth rate of the $m=0$ mode is never greater than zero as required by Cowling's theorem. The growth rate of the $m=1$ mode first decreases with increasing Rm. If the velocity field is a dynamo, that growth rate eventually increases and becomes positive for sufficiently large Rm.

Figures 1 and 2 are to be compared with Fig. 6 of Peffley *et al.* [2]. Applying an external field parallel or perpendicular to the symmetry axis excites predominantly $m=0$ or $m=1$ modes, respectively. It is thus perfectly compatible with the numerical results that the growth rate of modes excited by an axisymmetric external field increases as a function of Rm. No nonaxisymmetric features of the flow need to be invoked. The simulations are also in agreement with the fact that the growth rate of the fields excited by a pulse applied perpendicular to the axis of symmetry decreases as a function of Rm at low Rm. In order to achieve dynamo action, it will be necessary to reach high enough values of Rm so that this latter growth rate increases as a function of Rm and finally becomes positive. It is therefore impossible to deduce the critical Rm from Fig. 6 of Peffley *et al.* [2] for the flow realized in their experiment.

The growth rates for the $m=2$ and $m=3$ modes are not shown in Figs. 1 and 2 because they have much larger absolute values than for $m=0$ and 1. Consider, for instance, the free decay rates that correspond to the case Rm=0. The free decay rate for $m=0$ and 1 modes, λ_r, is equal to $\lambda_r=-7.99$. The free decay rates are $2.05\lambda_r$ for $m=2$ and $3.37\lambda_r$ for $m=3$. For Rm<80, the growth rate for the $m=2$ and 3 modes stays within 20% of the corresponding free decay rate.

Consider now the spatial dependence of the decay rate. If the experimental sequence of applying an external field, switching it off, and observing the field's decay, is repeated numerically for an axisymmetric velocity field, one finds of course the same decay rate at every position after an initial transient. However, the experimental flow is turbulent and only its time average is axisymmetric. If an axisymmetric external field is applied to such a flow, fields of all azimuthal wave numbers will be excited. The eigenmodes of any instantaneous realization of the turbulent velocity field is a superposition of modes of the axisymmetric flow field (1). During the decay of the magnetic field, the form of the eigenmodes apparently does not change significantly because otherwise the decay curves recorded by the Hall probes would not be nearly exponential. The correlation time of the eddies responsible for the mode mixing seems to be at least as long as the decay time of the magnetic field. In the present model, we will assume that the eigenmodes remain unmodified during the decay phase. The variations of the experimentally observed decay rates with Rm resembles the numerical results so that it is plausible to assume that every instantaneous eigenmode is close to an eigenmode of the time averaged flow and has nearly the same decay rate. In the following, no distinction will be made between a mode of a given

m of the axisymmetric flow and the corresponding mode of any realization of the turbulent flow that has small admixtures of different m. Because the mixed modes do not have a well defined symmetry, the axisymmetric external field excites a superposition of them. When the field is switched off, one observes a superposition of the decay rates of the various modes.

This decay has been followed during roughly three free decay times $1/|\lambda_r|$ in the experiment. The decay rates of the modes with $m=0,1$, and 2 are all close enough so that the eigenmodes cannot properly separate during the period of observation. As an example, let us consider the superposed decay of the most slowly decaying modes with $m=0$ and $m=2$ (whose decay rates differ approximately by a factor of 2). A semilogarithmic plot of $e^{-t}+e^{-2t}$ for $0\leqslant t\leqslant 3$ still looks reasonably close to a straight line. This will be true *a fortiori* for a superposition of $m=0$ and $m=1$ modes whose decay rates differ less. The decay curves observed experimentally are presumably not strictly exponential but a superposition of several exponentials.

In an axisymmetric flow, all eigenvalues are infinitely degenerate because every eigenmode can be rotated by an arbitrary angle about the axis of symmetry to yield another eigenmode with the same eigenvalue. Let us assume that the experimental flow is axisymmetric if averaged over time. Because of turbulent fluctuations, every realization of an external axisymmetric pulse excites a superposition of modes whose phase angles are randomly and equally distributed over the interval $[0,2\pi]$. A Hall probe on the exterior of the sodium container therefore registers a different apparent decay rate from pulse to pulse, but the histogram of these effective decay rates is the same for any Hall probe in an axisymmetric array as was used in the experiment.

This picture is further supported by the correlation measurements reported in Fig. 9 of Peffley *et al.* [2]. Decay rates measured by two Hall probes are decorrelated if the two probes are separated by roughly 90° but the correlation increases again for probes separated by 180°. Since the $m=1$ modes are the slowest decaying nonaxisymmetric modes, they will dominate the angular dependence of the decay rates. If an $m=1$ mode makes a significant contribution at the azimuth φ, it also makes a significant contribution at $\varphi+180°$ but negligible contributions at $\varphi\pm 90°$ because the amplitude of an $m=1$ mode varies in $e^{im\varphi}$.

This argument will now be made more precise. For the purpose of this paper, it is more important to reduce the number of adjustable parameters than to produce an accurate fit. We therefore limit ourselves to modes with $m=0$ and $m=1$. The temporal variation of the radial component of the magnetic field observed by a Hall probe at position φ is given by

$$B_r(\varphi,t)\propto e^{\lambda_0 t}+a\cos(\varphi-\varphi_1)e^{\lambda_1 t}. \tag{4}$$

λ_0 and λ_1 are the growth rates of the modes with $m=0$ and $m=1$ and φ_1 fixes the orientation of the $m=1$ mode with respect to the Hall probes. The coefficient a gives the amplitude of the $m=1$ mode in comparison with the $m=0$ mode.

We define p, the "apparent" growth rate of B_r as

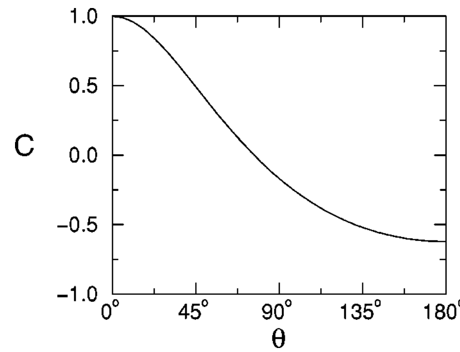

FIG. 3. $C(\theta)$ for $\lambda_0=-0.8$, $\lambda_1=-1.2$, and $a=0.9$. The values for λ_0 and λ_1 are taken from Fig. 6 of Ref. [2] for Rm around 40.

$$p(\varphi)=\ln\frac{B_r(\varphi,t=1)}{B_r(\varphi,t=0)}. \tag{5}$$

For simplicity, we assume that a is identical from pulse to pulse, but that φ_1 is randomly distributed. The ensemble average of the growth rate, $\langle p\rangle$, is then obtained by an average over φ_1 and is independent of φ:

$$\langle p\rangle=\frac{1}{2\pi}\int d\varphi_1 p=\frac{1}{2\pi}\int d\varphi_1\frac{e^{\lambda_0}+a\cos(\varphi-\varphi_1)e^{\lambda_1}}{1+a\cos(\varphi-\varphi_1)}. \tag{6}$$

With the same definition of the average $\langle\cdots\rangle$ one can compute the correlation function C:

$$C(\theta)=\frac{\langle[p(\varphi)-\langle p\rangle][p(\varphi-\theta)-\langle p\rangle]\rangle}{\langle[p(\varphi)-\langle p\rangle]^2\rangle}. \tag{7}$$

Figure 3 shows $C(\theta)$ for one set of parameters. As expected, the correlation first decreases as a function of θ and is zero for $\theta\approx 75°$. The absolute value of C then increases again and reaches a local maximum at $\theta=180°$. The entire curve is symmetric such that $C(\theta)=C(-\theta)$.

Figure 9 of Ref. [2] does not show $C(\theta)$ but a "regression coefficient." The regression coefficient carries, however, a similar information as $C(\theta)$. According to Fig. 9, correlation in the experiment is least for $\theta\approx 120°$ and the correlation increases only slightly in going from $\theta=120°$ to $\theta=180°$. There is little doubt that these quantitative aspects could be reproduced in the model by introducing more adjustable parameters through a probability distribution for a and modes with higher m. The quantity $C(\theta)$ could also be extracted from the experimental data and would allow a better comparison with the model presented here.

The numerical results suggest directions for future experimental work. It is obviously important to better characterize the time averaged velocity field so that decay rates predicted numerically on the basis of an experimentally determined flow field can be compared with experimental decay rates. In addition, the interpretation in terms of global magnetic eigenmodes advocated here can be tested as follows: Instead of determining a decay rate from a local measurement, one can use the Fourier transform with respect to φ of the signals of the entire array of Hall probes. One then obtains Fourier coefficients for different azimuthal wave numbers m that decay in time. These decay rates should correspond to the decay rates of modes with $m=0,1,\ldots$ irrespective of whether they have been excited by an axisymmetric field or a field perpendicular to the axis of the experiment. The ratio of the Fourier coefficients is also a measure of the coefficient a used above.

Peffley *et al.* [2] have also studied spectra of time series of magnetic field recorded at a fixed position. These spectra carry the imprint of the Kolmogorov spectrum of the velocity field. According to the picture delineated in this paper, the role of these small scale turbulent fluctuations for the dynamo effect at subcritical magnetic Reynolds numbers is reduced to mixing different modes that are decoupled in an axisymmetric time independent flow. The behavior of the magnetic field on the time scale of the free decay time and the length scale of the sodium container is still well described by kinematic dynamo theory applied to an axisymmetric laminar flow, which is actually reassuring for anyone involved in designing dynamo experiments.

[1] A. Tilgner, Phys. Earth Planet. Inter. **117**, 171 (2000).
[2] N. Peffley, A. Cawthorne, and D. Lathrop, Phys. Rev. E **61**, 5287 (2000).
[3] M. Dudley and R. James, Proc. R. Soc. London, Ser. A **425**, 407 (1989).
[4] P. Odier, J.-F. Pinton, and S. Fauve, Phys. Rev. E **58**, 7397 (1998).
[5] R. O'Connell, R. Kendrick, M. Nornberg, E. Spence, A. Bayliss, and C. Forest, in *Dynamo and Dynamics, A Mathematical Challenge*, edited by P. Chossat, D. Armbruster, and I. Oprea (Kluwer, Dordrecht, The Netherlands, 2001).
[6] D. Sweet, E. Ott, J. Finn, T. Antonsen, Jr., and D. Lathrop, Phys. Rev. E **63**, 066211 (2001).
[7] A. Tilgner, Phys. Lett. A **226**, 75 (1997).